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Resonance poles and width distribution for time-reversal transport through mesoscopic open billiards

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A number of resonance poles are computed for time-reversal ballistic transport through chaotic and integrable mesoscopic billiards coupled to a pair of single-channel leads. The width distribution of resonances is rigorously compared with the random-matrix-theory prediction, which has been recently obtained for time-reversal chaotic open systems with overlapping resonances. In the case of chaotic open billiards, the distribution functions show good agreement with the random-matrix-theory prediction in all ranges of the width. In the case of integrable open billiards, however, there exist some deviations and the agreement is perceived only for the tail of the distribution functions. This is understood quantitatively in terms of classical decay-time distributions.

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The study of chaotic open billiards is interesting because fully chaotic and integrable motion of noninteracting particles inside ballistic cavities can be realized, simply depending upon the design of its boundary. It is intriguing to study the effects of the underlying classical dynamics on quantum transport through such a system, in connection with an application to mesoscopic devices.

In general, quantum transport coefficients, such as conductance, in open billiards show ample oscillations as a function of external parameters, see, e.g., [1,2]. This originates from a sequential overlap of resonances in the cavity region lying inside the billiard. In order to better understand resonance structures, it is necessary to identify poles and analyze their properties in detail. Gaspard and Rice, in their pioneering papers, obtained the location of the exact poles in the complex wave number plane for the scattering of a point particle from a chaotic *convex* repeller consisting of three hard discs in a plane [3]. To the best of the author's knowledge, only a few reports on the poles in *concave* open billiards have been recently published for integrable cases [4–7] and a pseudointegrable case [8]. In the latter, the authors numerically found resonances by using a simple Hamiltonian model describing a time-reversal rectangular microwave resonator perturbed with an attached antenna, and attempted to compare the width distribution of the isolated resonances with a random-matrix-theory prediction valid for the regime of *overlapping* resonances for *chaotic* open systems with *broken* time-reversal symmetry [9]. Thus we can say that our knowledge about statistical properties of resonances for chaotic open billiards is far from satisfactory, in connection with random-matrix-theory predictions.

In this study, we numerically obtain a number of resonance poles for chaotic and integrable open billiards. Then, we rigorously compare the width distribution of the resonances with the exact analytical formula derived recently in the regime of overlapping resonances for chaotic open systems with time-reversal symmetry [10]. Although the pole parameterization of the scattering matrix cannot be uniquely determined by measuring cross sections in experiments when the resonances become more and more overlapped [11,12], this study merely proposes analyses for future research.

The quantum open billiard we wish to study consists of a two-dimentional cavity coupled to continua by attached leads that altogether support M equivalent open channels. In a weakly [13] and *imperfectly* open limit, there have been a number of measurements on resonances of microwave cavities and the system, in general, shows isolated resonances. They can be fitted by the Breit-Wigner formula [14] and simply explained based on the eigenmodes and eigenfunctions of a closed counterpart of the cavity [15–19] (plus perturbations by antennas [8,20]). When the cavity is fully chaotic, the width distribution of such resonances is quite generally expected to follow the so-called χ^2 distribution with parameter $\nu = M (\nu = 2M)$ for systems with preserved (broken) time-reversal symmetry. The case $\nu = 1$ is known as the Porter-Thomas distribution [21] and has been shown to be in agreement with experimental data [22]. On the other hand, when the cavity is weakly but *perfectly* open, we commonly see inevitably overlapping resonances in quantum transport. This is the case in this paper.

We consider a Bunimovich stadium [23] and its deformations as chaotic billiards, and a circle and a square as integrable billiards, respectively. The billiard is coupled to a pair of leads with a common width d and their orientations are not straightforward to avoid direct transmission; see Fig. 1. The stadium billiard is characterized by the radius of a semicircle a and the half-length of a straight section l. The aspect ratio $\sigma \equiv l/a$ is continuously tunable, keeping the area of the billiard $A = \pi a^2 + 4al$ fixed, which ensures the same degree of resonance density for each billiard. For a closed stadium, the maximum Lyapunov exponent vanishes in the integrable limit ($\sigma = 0$) and reaches its maximum at the fully chaotic limit ($\sigma = 1$) [24]. The limit $\sigma = 0$ corresponds to the circle billiard. In the following, we refer to the limit $\sigma = 1$ as the stadium and to the cases $\sigma = 0.99$, 0.995, 1.005, and 1.01 as its deformations, respectively. We use scaled lengths such that the enclosed area for each of the billiards is $A = \pi + 4$ (i.e., $l = a = 1$ for the stadium). Therefore the length of the sides of the square billiard is $\sqrt{\pi + 4}$. We choose a small lead

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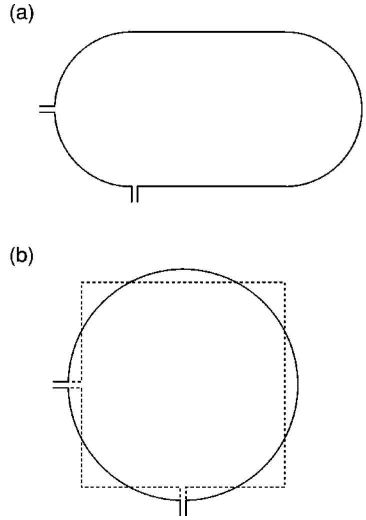

FIG. 1. Geometry of open billiards. (a) Stadium. (b) Circle (solid lines) and square (dotted lines).

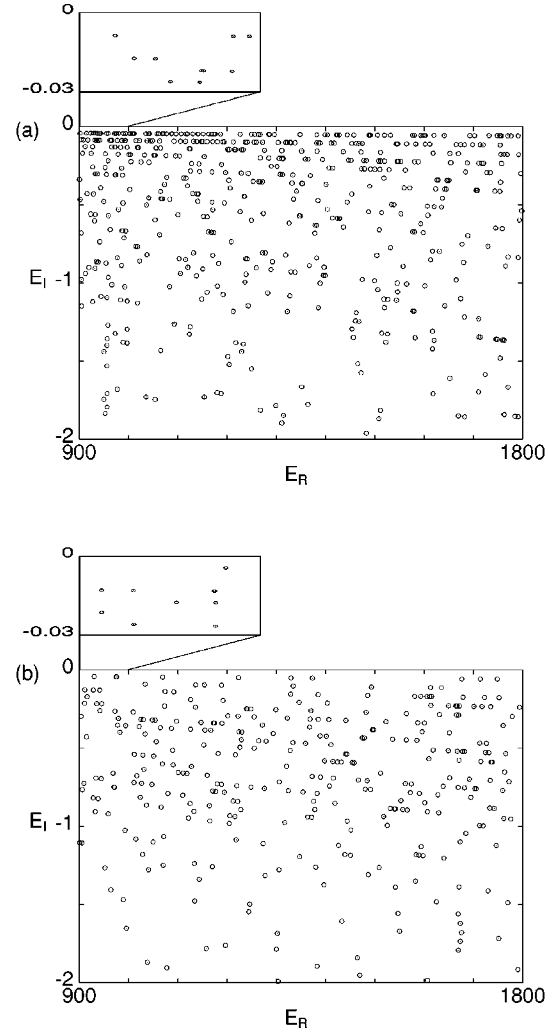

FIG. 2. Location of resonance poles in the complex energy plane. The partial magnification shows a more accurate plot of a small region near the real axis. (a) Open stadium billiard. (b) Open circle billiard.

width as $d=0.08$, which corresponds to a weakly open billiard where a particle entering through one lead dwells inside the cavity region for a long time before exiting. Assuming the ergodic billiards, the estimated decay length of a classical path-length spectrum is $L_d=140$ [25], which corresponds to $L_d/\sqrt{A}\simeq 52$ bounces inside the cavity. On the other hand, the mixing length scale in the stadium billiard is expected to be $L_m<100$ [24]. Thus $L_d>L_m$. Therefore the particle acquires chaotic and nonchaotic features through multiple scattering with specular reflections on the boundary of the cavity, and this will affect its transport properties.

In quantum dynamics, the dc current passes through the leads. We choose the energy of the incoming wave so that only the first transverse mode in the leads is open. We solve the time-independent Schrödinger equation under Dirichlet boundary conditions based on the plane-wave-expansion method [26], giving reflection and transmission amplitudes as a function of the energy. The resonances are due to quasibound states of the open billiard and they are identified with the poles of the corresponding scattering matrix $\mathbf{S}(E)$ occurring at complex energies $E^\alpha=E_R^\alpha+iE_I^\alpha$ ($E_I^\alpha<0$) for the αth resonance. They are numerically obtained as the singular points of $\mathbf{S}(E)$ by scanning E on the meshed lower complex plane. The positions and widths of the resonance states are given by E_R^α and $\Gamma^\alpha(=2|E_I^\alpha|)$, respectively. In the following, we will assume $\mathbf{S}(E)$ to be a simple pole, $\sim(E-E^\alpha)^{-1}$, for E close to a resonance energy, and choose both $\hbar$ and the mass of particle μ as unity for simplicity.

In the energy range investigated in this paper, we can observe a number of overlapping resonances, resulting in ample fluctuations in the transmission probability T as a function of the real energy E of particles. To understand the fluctuations in detail, the location of resonance poles is computed in the complex energy plane, and presented in part in Fig. 2 for the open stadium and circle billiards. Each point corresponds to an isolated resonance state. The quantum decay length for the deepest poles shown in Fig. 2 is estimated using the lifetime $\tau_d^{gm}\sim\hbar/\Gamma$ and the velocity $v=\sqrt{2E_R/\mu}$ as $L_d^{qm}\equiv\tau_d^{gm}v\simeq 11-15$, which corresponds to $4-6$ bounces inside the cavity. We see that the resonance poles are placed irregularly in the complex energy plane in both cases, except a tendency to accumulate themselves on lines. This linear structure simply comes from the coarseness of numerical data in the complex plane. We should also mention that some of the poles closest to the real axis or to each other (i.e., almost degenerate) in the range $E_I\gtrsim-0.3$ ($E_I\gtrsim-0.035$ for the open square billiard, not shown here) are missing because of a simple numerical reason. This means that many of the sharpest oscillations of $T(E)$ are not identified (see the more accurate plot in Fig. 2). The mean resonance spacing Δ calculated from our data is 1.96, 2.08, and 2.58 for the stadium and its deformations, the circle, and the square, respectively. According to a simple comparison between Weyl's density of states ($\sim\mu A/(2\pi\hbar^2)$) and $1/\Delta$, about 55% of the eigenmodes are missing in the case of the stadium and its deformations, about 58% in the case of the circle, and about 66% in the case of the square. As we see in Fig. 2, the open circle billiard has a smaller number of sharp resonances than the others (including the open square billiard not shown here), which can be understood from the following. When the cavity has a rotational symmetry, the coupling of wave functions, ψ_{lead} and ψ_{cavity}, can easily induce rotational change of ψ_{cavity} to try to hold parity conservation law with each other locally at the openings, so that resonances in such an open billiard, in general, tend to be broad [27].

Once we obtain the location of poles in our systems, we calculate the width distribution of resonances, where we use poles in the energy range $770 \leqslant E_R \leqslant 1800$ (including 490 $\sim$565 poles), $800 \leqslant E_R \leqslant 1800$ (including 479 poles), and $860 \leqslant E_R \leqslant 2450$ (including 615 poles) for the stadium and its deformations, the circle, and the square, respectively. On the other hand, the analytical expression of the width distribution for the overlapping resonances was obtained employing random-matrix theory first for chaotic open systems with broken time-reversal symmetry [9], and quite recently for those with preserved time-reversal symmetry [10]. In the latter case, the distribution function of scaled resonance widths $y=(\pi E_I/\Delta)$ for perfectly open systems with $M=2$ is given in Ref. [10] by $P(y<0)=(2y+1+(2y-1)\,e^{4y})/(4y^3)$. When we calculate the width distribution and compare it with this analytical expression, we find interesting features in its behavior, as we see in Fig. 3. In the case of chaotic open billiards, $P(\Gamma)$ follows Γ^{-2} asymptotically for $\Gamma \gg \Delta$ while it asymptotes to a constant for $\Gamma \ll \Delta$, as is predicted in [10]. We still notice small deviations for $\log_{10}\Gamma \lesssim -0.2$ and $\log_{10}\Gamma \gtrsim 0.5$. As previously mentioned, many of the resonances for $\Gamma = 2|E_I| \lesssim 0.6$ were not identified in the calculation, although they should also be included to compare with the analytical result. The difference of $\log_{10}P(\Gamma)$ between the numerical and the analytical results for that region is roughly estimated as -0.25 in Fig. 3(a). This corresponds to a 44% decrease of $P(\Gamma)$ and almost agrees with the estimation of the number of missing eigenmodes, as is mentioned above. Therefore we expect the decrease of $P(\Gamma)$ for $\log_{10}\Gamma \lesssim -0.2$ in the numerical result, and relatively the increase by a constant for a larger Γ region as a result of the probability conservation of $P(\Gamma)$. (This argument also holds true in the case of integrable open billiards.) Taking this into account, it seems reasonable to say that $P(\Gamma)$, computed for the chaotic open billiards, is in good agreement with the analytical prediction by random-matrix theory in all the ranges of width Γ shown in Fig. 3. In the case of integrable open billiards, $P(\Gamma)$ also seems to follow Γ^{-2} asymptotically for $\Gamma \gg \Delta$; however, for the smaller Γ region the deviations from the analytical expression by the random-matrix theory are prominent and the behavior no longer shows any universality [see Figs. 3(b) and 3(c)]. This nonuniversal behavior is typical for integrable open billiards and especially sensitive to the symmetry restriction of the system. For example, in the case of the open circle billiard, the rotational symmetry of the cavity boundary results in a considerable reduction of the number of resonances with small Γ, as mentioned above.

One may show that the physical meaning of the power-law tail, $\sim\Gamma^{-2}$, turns out to be due to classical processes of exponential escape typical for fully chaotic systems [28], as was pointed out in [9]. In weakly open billiards, however, such an exponential escape is observed not only for chaotic cases [2], but also for integrable cases [29] in a time scale corresponding to a path length up to at least a few hundreds. Using a relation $t \approx \hbar/\Gamma = -\pi\hbar/(2y\Delta)$, this escape time distribution $P(t)\,dt = (1/\tau_d)\exp(-t/\tau_d)\,dt$ with a lifetime τ_d can be transformed into a *classical* width distribution [30]:

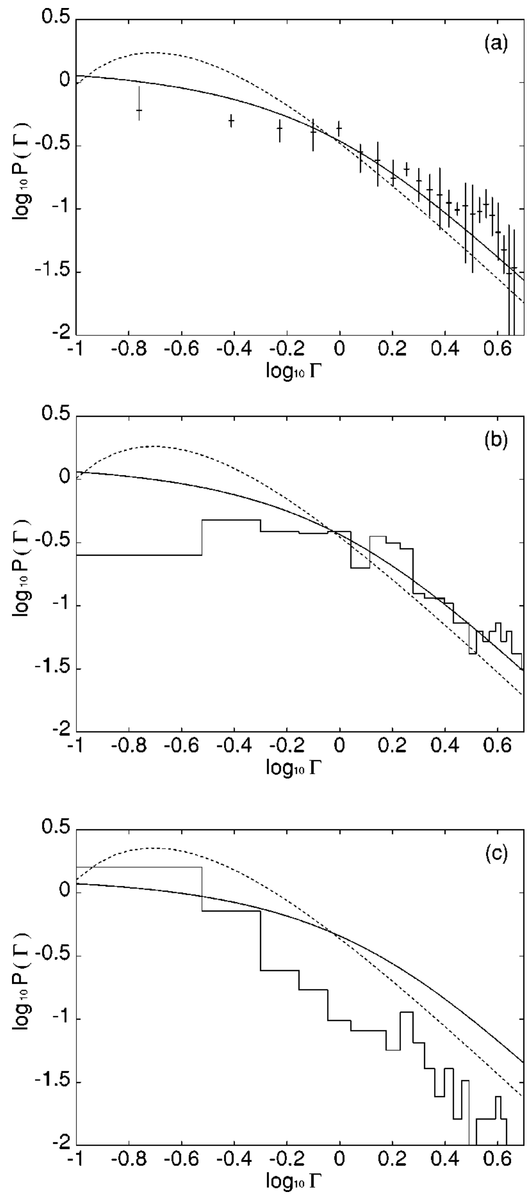

FIG. 3. Width distribution of resonances (cross bars or open bins), theoretical prediction by Sommers *et al.* for time-reversal fully chaotic systems with $M=2$ (solid line), and classical estimation (dotted line). (a) Open stadium billiard and its four deformations. Horizontal bar indicates the average value. (b) Open circle billiard. (c) Open square billiard.

$$P_{cl}(y<0) = \frac{c}{y^2}\exp\left(\frac{c}{y}\right), \tag{1}$$

where $c = \pi\Gamma_{cl}/(2\Delta)$ and $\Gamma_{cl} \equiv \hbar/\tau_d = (\hbar/L_d)\sqrt{2E_R/\mu}$. For $|y| \gg 1$, $P_{cl}(y) \propto y^{-2}$. In Fig. 3, Eq. (1) is plotted with a dotted line for a typical energy $E_R = 1500$ in the data. We see that it is successfully compared with the power-law tail of the distribution function in both chaotic and integrable cases; however, a failure is evident for $\Gamma \to 0$, where $P_{cl}(y \to -0) = 0$, while a purely quantum effect in resonances for the weak coupling limit dominates with strongly diminishing Γ, leading to $P(\Gamma \to 0) \sim \text{const.}$

Finally, although the billiard is connected to perfect leads, the coupling to open channels is practically *not perfect* owing to a diffraction effect at the corners of the openings [31,32]: It is totally imperfect at the energy right after the opening of a new channel in the lead, eventually leading to the perfect coupling realized only in the high-energy limit for the same channel. In our systems, however, the diffraction effect on the transmission through an opening is numerically estimated to be about 10% for $E=916$ and less than 1% for $E\geqslant 1400$ [32]. Therefore it may not strongly affect the overall statistical features of the resonance distribution in the energy range that we adopted in our calculations.

In conclusion, we showed that our numerical results for the width distribution of overlapping resonances in the case of time-reversal chaotic open billiards support the random-matrix-theory prediction, serving as its good numerical tests. In the case of integrable open billiards, however, there exist some deviations, and the agreement is perceived only for the tail of the distribution functions. This is understood quantitatively in terms of classical decay-time distributions.

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